1 Introduction

1.1 Differential equations

Let us consider a real function of one real variable $y = y(x)$. The following equations fall under the category of ODEs:

$$y' = y, \quad y'' - y = 0. \quad y'' - y^2 = 0.$$

The solution to the first equation is given by $y = ce^x$, where c represents an arbitrary constant. To pick out a specific solution, we would have to introduce an initial condition into the ODE. For instance, if we stipulate that $y(0) = 1$, then c becomes equal to 1 and the solution would be $y(x) = e^x$.

The second equation is a second-order linear ODE. The standard form of such equations is expressed with three known functions, p, q and r, such that:

$$y'' + p(x)y' + q(x)y = r(x).$$

Then the ODE is said to be homogeneous if $r = 0$ and inhomogeneous otherwise. The total solution y of the ODE 1.1 is the sum of a homogeneous solution y_h (solution of the ODE considering $r = 0$) and a particular solution y_p ($y = y_h + y_p$). The particular solution y_p can be obtained, for instance, using the variation of the constant.

A linear homogeneous second-order ODE with constant coefficients, $a,b,c \in \mathbb{R}$, can be written as

$$ay'' + by' + cy = 0.$$

The solution of such equation depends on those of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

We distinguish the following cases.

1. Two distinct real roots λ_1 and λ_2. The solution is $y_h = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$, $C_1, C_2 \in \mathbb{R}$.
2. Double real root $\lambda = -\dfrac{b}{2a}$. The solution is $y_h = [C_1 + C_2x]e^{\lambda x}$.
3. Complex conjugate roots $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$.

$$y_h = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right].$$

In contrast, Partial Differential Equations (PDEs) encompass partial derivatives due to their involvement with more than one independent variable, such as x, y, and others.

Notations

We commonly represent the partial derivatives of a function u with respect to independent variables x, y, using various notations, including:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial y \partial x}, \quad u_x, \quad u_{xy}, \ \ldots$$

For instance, consider a function $u = u(x, y)$ and the following PDEs:

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

Solving the first equation is relatively straightforward, and it can be verified that the solutions include:

$$u = c, \quad u = y, \quad \text{and} \quad u = y^2 + \ln|y|.$$

In fact, the most general solution takes the form $u = f(y)$, where f is an arbitrary function. Similarly, the general solution for the second equation has the form $u(x) = g(x)$. The general solution of the third equation is given by $u = f(x - y)$, where f is an arbitrary function of its argument. A nuanced distinction between solutions of ODEs and solutions of PDEs is that ODE solutions incorporate constants of integration, while PDE solutions encompass functions of integration.

1.2 Basic concepts

Definition 1.1 (PDE). A **partial differential equation** (**PDE**) is an equation involving one or more partial derivatives of an (unknown) function, that depends on two or more independent variables. The **order** of the differential equation is the highest partial derivative that appears in the equation. More formally, a PDE can be written in the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots) = 0, \tag{1.1}$$

where F is a given function of the independent variables $\mathbf{x} = (x_1, \ldots, x_n)$; of the unknown function $u = u(\mathbf{x})$ and of a finite number of its partial derivatives. The PDE (1.1) can also be written in the form

$$\mathcal{L}u = f, \tag{1.2}$$

where where $f = f(\mathbf{x})$ and $\mathcal{L}$ is an operator that contains all the operations (differentiation, multiplication, composition, etc.) that act on u. Without loss of generality, we consider $n = 2$ in th remaining.

Definition 1.2 (Classification)**.**

- We say that a PDE is **linear** if $\mathcal{L}$ is linear and **nonlinear** otherwise.
- A PDE is called **semilinear** when it is linear in the leading (highest-order) terms.
- A PDE of order m is said to be **quasi-linear** if it is linear in the derivatives of order m with coefficients that depend on $\mathbf{x}$, u and the derivatives of order $< m$.
- Linear equations can further be classified as **homogeneous** when $f = 0$ and **non-homogeneous** otherwise.

Proposition 1.2.1.

- *The superposition principle: any linear combination of two solutions of a linear homogeneous PDE is also a solution.*
- *For a linear PDE, the sum of a homogeneous solution and an inhomogeneous solution is also an inhomogeneous solution.*

Example 1.1 (General forms of first order PDEs)**.**

- Linear PDE: $a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$
- Semi-linear PDE: $a(x, y)u_x + b(x, y)u_y = d(x, y, u)$
- Quasi-linear PDE: $a(x, y, u)u_x + b(x, y, u)u_y = d(x, y, u)$

Example 1.2.

- The heat equation: $u_t - u_{xx} = x$ (linear non-homogeneous second-order, $\mathcal{L} = \partial_t - \partial^2_{xx}$).
- The heat equation: $\frac{\partial u}{\partial t} - c^2\Delta u = 0$ (second-order, linear, homogeneous, $\mathcal{L} = \partial_t - c^2\Delta$).
- The wave equation: $u_{tt} - u_{xx} - x^3 = 0$ (non-homogeneous linear second order, $\mathcal{L} = \partial^2_{tt} - \partial^2_{xx}$.
- The wave equation: $\frac{\partial u^2}{\partial t^2} - c^2\Delta u = 0$ (second-order, linear, homogeneous, $\mathcal{L} = \partial^2_{tt} - c^2\Delta$).
- The Laplace equation: $\Delta u = 0$ (second-order, linear, homogeneous, $\mathcal{L} = \Delta$).
- The Poisson equation: $\Delta u = f$ (second-order, linear, non-homogeneous if $f \neq 0$).
- The transport (convection) equation: $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ (first-order, linear, homogeneous)
- The Burger's equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ (first-order, quasilinear, homogeneous, $\mathcal{L} = \partial_t + u\partial_u$).
- The Korteweg de Vries equation (**KdV**): $u_t + 6uu_x + u_{xxx} = 0$ (third-order, semi-linear, homogeneous, $\mathcal{L} = \partial_t + u\partial_u + \partial^3_{xxx}$).

Definition 1.3. A solution of a PDE for a function $u = u(x, y)$ can be visualized as a surface in the three-dimensional space (x, y, u), called **integral surface**.

Definition 1.4 (Classification of PDEs problems.)**.** Consider a given PDE on a region Q.

1. **Cauchy** or **initial-value** problem: some additional initial-values for the unknown function and its derivatives are given on some subsets of Q.

2. **Boundary value problem**: some additional boundary conditions for the unknown function and its derivatives are given on ∂Q.

3. **Mixed type problem**: some additional initial-values and boundary conditions for the unknown function and its derivatives are given.

Definition 1.5. A problem with a PDE is **well-posed** in a class of functions $\mathcal{C}$, if the following three conditions are satisfied.

1. there exists a solution in $\mathcal{C}$;

2. the solution is unique;

3. the solution is continuously dependent on the given conditions, e.g., initial-values, boundary conditions, coefficients, etc.

Example 1.3. Show that $u(x, y) = y^2 + x$ is the solution of the Cauchy problem on $\mathbb{R}^2$:

$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad u(0, y) = y^2 \ (y \in \mathbb{R}).$$

1.3 Modeling with PDEs

PDEs are widely recognized as "Mathematical Physics Equations", with applications spanning diverse fields such as acoustics, optics, elasticity, hydrodynamics, aerodynamics, electromagnetism, quantum mechanics and chemistry, seismology, chemical kinetics. Furthermore, PDEs arise in unexpected arenas like economics, financial mathematics, social sciences, and image processing. In the following, we delve into the origins of specific PDEs.

1.3.1 Conservation Laws

Studying natural processes, we observe two main tendencies: the tendency to achieve a certain balance between causes and consequences, or the tendency to break this balance. We usually use some law or principle that expresses such a balance between the so-called state quantities u and flow quantities q, and their spatial and time changes. Here are some examples.

- State quantities: density, pressure, temperature, entropy, ...

- Flow quantities: velocity, momentum, tension, heat flux, ...

Consider a function $u = u(x, t)$ representing the density or the concentration of a physical quantity m (e.g. a mass in a region, heat in a metal bar, traffic on a highway) and $q(u)$ is its flux function (the rate of change of m). If we consider a control interval (x_1, x_2), the integral

$$\int_{x_1}^{x_2} u(x, t)\, dx$$

gives the amount of m contained inside (x_1, x_2) at time t. Without sources or sinks, the rate of change of m inside (x_1, x_2) is determined by the net flux through the end points:

$$\frac{\partial}{\partial x} \int_{x_1}^{x_2} u(x, t) dx = -q(u(x_2, t)) + q(u(x_1, t)),$$

and thus, for smooth functions u and q, we get

$$\int_{x_1}^{x_2} [u_t(x, t) dx + q(u(x, t))_x] dx = 0,$$

which, due to the arbitrariness of the interval (x_1, x_2), implies

$$u_t + q(u)_x = 0, \quad x \in \mathbb{R}, t > 0. \tag{1.3}$$

Equation 1.3 is a **scalar conservation law** that gives a link between density and flux. At this point we have to establish a constitutive relation for q that describes the flux function.

For a higher dimension, $q(u)$ becomes a vector-valued function. With a scalar source function $f = f(x, t)$, the conservation law writes:

$$u_t(x, t) + \nabla \cdot q(u)(x, t) = f(x, t), \quad x \in \mathbb{R}^N, t > 0. \tag{1.4}$$

When reaching a steady or stationary state, all quantities become time-independent and thus, simplified versions of the conservation laws are obtained.

1.3.2 Constitutive Laws

The conservation law represents one equation for two unknown functions, the state and the flow quantities. To create a solvable mathematical model, we need another relation between these functions. Such relations are usually based on the generalization of experimental observations and depend on the properties of the particular medium or material. A typical example of the constitutive law from the elasticity theory is Hook's law, which states that, for relatively small deformations of an object, the displacement or size of the deformation (state quantity) is directly proportional to the deforming force or tension (flow quantity). Some constitutive laws will be given in the next section.

1.3.3 Basic Models

Advection (Transport) equation

Advection is the transport of a substance such as mass, energy, or concentration, or a conserved scalar, by a known velocity field (of a fluid). The advection equation is a fundamental equation used in various fields, including fluid dynamics, meteorology, and transport phenomena. It is a first-order PDE and can be written in one dimension as:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where $u(x,t)$ is the quantity being transported (e.g., concentration of a substance or a temperature), t is time, x is the spatial coordinate, and $c(x,t)$ is the velocity.

Example 1.4. Suppose a certain amount of a chemical spills into a river and flows downstream. Let us suppose that the concentration is given by $u = u(x,t)$ and that the river flows with constant speed c. If we assume that the chemical does not diffuse, then the flux is proportional to the quantity u: $q(x,t) = cu(x,t)$ (which is the constitutive law). Consequently, using the scalar conservation law, we obtain

$$u_t + cu_x = 0.$$

As we will show later, the solution of (1.11) is a function

$$u(x,t) = F(x - ct),$$

where F is an arbitrary differentiable function. Such a solution is called the right traveling wave, since its graph at a given time t is the graph of the function $F(x)$ shifted to the right by the value ct. Thus, with growing time, the profile $F(x)$ is moving without changes to the right at the speed c (see Figure 1.1).

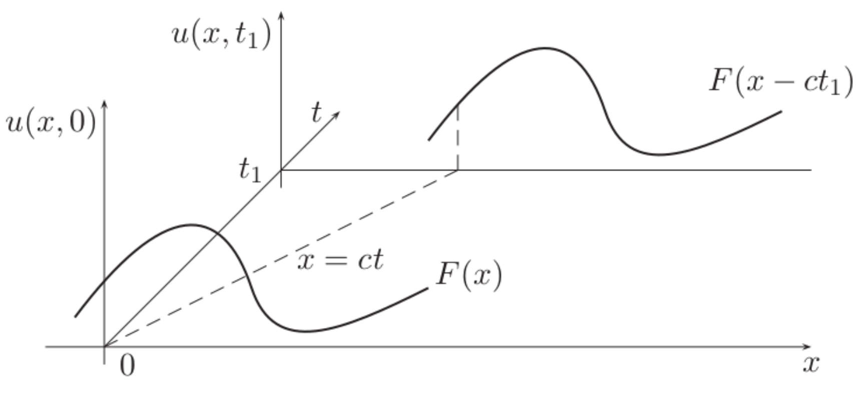

Figure 1.1: Traveling wave

Wave equation (vibrating strings). The wave equation is a fundamental PDE that describes how waves propagate through a medium. It is used to model a wide range of wave phenomena, including acoustic waves, electromagnetic waves, and mechanical waves. The one-dimensional wave equation, which represents wave propagation along a single spatial dimension x is given by:

$$\frac{\partial u^2}{\partial t^2} = c^2 \frac{\partial u^2}{\partial x^2}. \tag{1.5}$$

Example 1.5. To derive the wave equation in one spacial dimension, we consider an elastic string with a constant mass density ρ that undergoes small amplitude vertical vibrations. An infinitesimal element located between x_1 and x_2, with $\Delta x = x_2 - x_1$ is shown in Fig. 1.2.

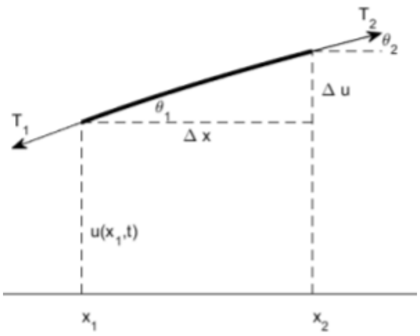

Figure 1.2: Derivation of the wave equation.

We define $u(x, t)$ to be the vertical displacement of the string from the x-axis at position x and time t, T the tension of the string **assumed to be constant**, and θ the angle between the string and the horizontal line. We assume that the string element **only accelerates vertically** (horizontal forces balance) and that the **gravity is negligible** compared to T.

We apply the Newton's law of motion ($m\,x''(t) = \vec{F}$) for the vertical acceleration of our infinitesimal string element (u_{tt}). The "infinitesimal" mass can be obtained as

$$m = \rho\sqrt{\Delta x^2 + \Delta u^2} \approx \rho \Delta x \sqrt{1 + u_x^2}$$

because $u_x \approx \Delta u / \Delta x$. Consequently,

$$\rho \Delta x \sqrt{1 + u_x^2}\; u_{tt} = T \sin\theta_2 - T \sin\theta_1.$$

With the **assumption of small vibrations** ($\Delta u << \Delta x$ or $u_x << 1$), we have

$$\sin\theta_i \approx u_x(x_i, t), \;\; \sqrt{1 + u_x^2} \approx 1.$$

Therefore, we get

$$\rho\, \Delta x\, u_{tt} = T\, \left(u_x(x_2, t) - u_x(x_1, t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = T/\rho$.

Diffusion equation. Diffusion is a fundamental physical process that describes the movement of particles or molecules in a fluid or a solid, from regions of higher concentration to regions of lower concentration, to spread out and mix with neighboring particles, resulting in the equalization of the concentration distribution over time (at equilibrium). Diffusion is governed by the so-called heat equation:

$$\frac{\partial u}{\partial t} = \Delta u. \tag{1.6}$$

Example 1.6. Consider a rod of length L, cross section area A and density $\rho(x)$ (Fig. 1.3).

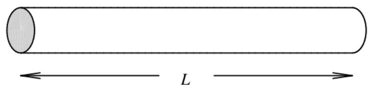

Figure 1.3: The typical cross-section

Given the specific heat $c(x)$ and the temperature $u(x,t)$, we define the total heat H as

$$H(x,t) = \int_0^L c(x)\rho(x)u(x,t)Adx. \tag{1.7}$$

The amount of thermal energy per unit time flowing to the right per unit surface area is called "heat flux" defined by $\Phi(x,t)$. If heat is generated within the rod and its heat density $Q(x,t)$ is given, then the total heat generated is

$$\int_0^L Q(x,t)Adt. \tag{1.8}$$

The conservation of heat energy states that: the rate of change of total heat = heat flowing across boundaries + heat energy generated inside. Mathematically, we can write

$$\frac{\partial H}{\partial t} = (\Phi(0,t) - \Phi(L,t))\, A + \int_0^L Q(x,t)Adt. \tag{1.9}$$

Consequently,

$$\int_0^L c(x)\rho(x)\frac{\partial u}{\partial t}(x,t)Adx = -\int_0^L \frac{\partial \Phi}{\partial x}Adx + \int_0^L Q(x,t)Adt, \tag{1.10}$$

which gives

$$\int_0^L \left(c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial \Phi}{\partial x} - Q(x,t) \right) Adx = 0, \tag{1.11}$$

Since this applies to any length L, then

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = -\frac{\partial \Phi}{\partial x} + Q(x,t) \tag{1.12}$$

Fourier suggested the following form for heat flux:

$$\Phi = -k\frac{\partial u}{\partial x} \tag{1.13}$$

where k is the thermal conductivity. This then gives

$$\frac{\partial u}{\partial t} = D\frac{\partial u^2}{\partial x^2} + f(x,t), \tag{1.14}$$

where $D = k/\rho c$, is the coefficient of diffusion, and $f = Q/\rho c$, is the source term.

Laplace equation The solutions of Laplace's equation are used in many important fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. A function satisfying Laplace's equation in a region of the plane (or 3-space) is said to be harmonic.

Example 1.7. Consider the example 1.6 of heat diffusion in 2D case. Suppose that $f(x,t)$ vanishes after some time. When heat diffuses and reaches an equilibrium (no variations over time), the heat diffusion equation becomes *stationary equations*, the simplest of which is Laplace equation

$$\Delta u(x,t) = 0.$$