Exercise 1.1. Find the general solution of the following equations:

1. $y'' - 8y' + 16y = 0$.
2. $y'' + k^2 y = 0$, for a constant k > 0.
3. $y'' - 6y' + 13y = 0, y(0) = 0, y'(0) = 10$.

Exercise 1.2. Which of the following operators are linear?

- $\mathcal{L}_1 u = u_x + x u_y$; $\quad \mathcal{L}_2 u = u_x + u u_y$; $\quad \mathcal{L}_3 u = u_x + u_y^2$; $\quad \mathcal{L}_4 u = u_x + u_y + 1$
- $\mathcal{L}_5 u = \sqrt{1+x^2}(\cos y) u_x + u_{yxy} - [\arctan(x/y)] u$

Exercise 1.3. For each of the following equations, state the order and whether it is linear/nonlinear/semilinear, quasilinear, homogeneous; provide reasons.

1. $u_t - u_{xx} + 1 = 0$; $\ u_t - u_{xx} + xu = 0$; $\ u_t - u_{xxt} + u u_x = 0$; $\ u_{tt} - u_{xx} + x^2 = 0$
2. $i u_t - u_{xx} + u/x = 0$; $\ u_x + e^y u_y = 0$; $\ x u_x + y u_y = u$; $\ x u_x + y u_y = u^2$
3. $u_x + (x+y) u_y = xy$; $\ u u_x + u_y = 0$; $\ x u_x^2 + y u_y^2 = 2$.
4. Shock wave: $u_x + u u_y = 0$; Wave with interaction: $u_{tt} - u_{xx} + u^3 = 0$; Dispersive wave: $u_t + u u_x + u_{xxx} = 0$

Exercise 1.4.

- Verify by direct substitution that $u_n(x, y) = \sin nx \sinh ny$ is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.
- Verify that $u(x, y) = f(x) g(y)$ is a solution of the PDE $u u_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.
- Show that $u_1(x, y) = x$ and $u_2(x, y) = x^2 - y^2$ are solutions to Laplace's equation. How can you combine them to create a new solution?
- Show that the soliton

  $$h(x, t) = 2\alpha^2 \text{sech}\left(\alpha(x - 4\alpha^2 t)\right)$$

  satisfies the the Korteweg-deVries equation,

  $$h_t + 6 h h_x = h_{xxx}$$

- The PDE
$$v_t - 6v^2 v_x + v_{xxx} = 0$$
is known as the modified Korteweg de Vries (mKdV) equation. Show that if v is a solution of the mKdV, then
$$u = v_x - v^2$$
is a solution of the KdV
$$u_t + 6uu_x + u_{xxx} = 0.$$

Exercise 1.5. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^2$ with the boundary conditions $u(x, 0) = 0$.

1. Show that
$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny$$
are solutions of the problem.
2. Compute the limit of u_n when $n \to \infty$.
3. Is the problem defined by the PDE and the boundary condition stable?

Exercise 1.6. Consider the traffic flow in a highway as shown on the next figure.

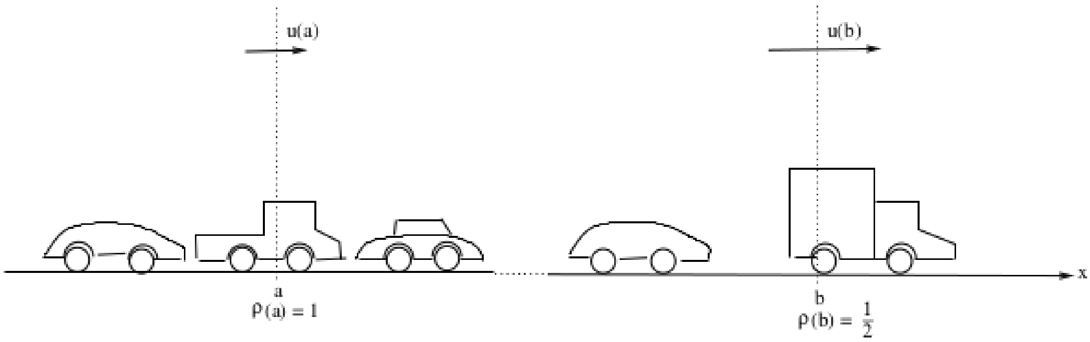

Figure 1.5: Traffic flow in a highway

Let $\rho(x, t)$ be the density of cars and $u(x, t)$ their velocity. We would like the express a PDE that describes the density function ρ.

1. How can you express the quantity of vehicles between a and b and its variation over time?
2. Express the difference between the traffic inflow (at $x = a$) and outflow ($x = b$) in terms of ρu.
3. Combine the two previous relations to deduce the equation $\rho_t + (u\rho)_x = 0$.

4. Assume that $u = 1 - \rho$. Motivate this choice then deduce a nonlinear convection equation of ρ.

5. Assume that $u = c - \epsilon(\rho'/\rho)$. Motivate this choice then deduce a convection-diffusion equation of ρ.

Exercise 1.7. Consider a smooth surface in $\mathbb{R}^{n+1}$ representing the graph of a function $x_{n+1} = u(x_1, ..., x_n)$ defined on a bounded open set Ω in $\mathbb{R}^n$. Assuming that u is sufficiently smooth, the area of the surface is given by the nonlinear functional

$$\mathcal{A}(u) = \int_\Omega \left(1 + |\nabla u|^2\right)^{1/2} dx_1 ... dx_n.$$

The minimal surface problem is the problem of minimizing $\mathcal{A}(u)$ subject to a prescribed boundary condition $u = g$ on the boundary of Ω. A classical result from the calculus of variations asserts that if u is a minimiser of $\mathcal{A}(u)$, then it satisfies the Euler-Lagrange equation:

$$\nabla \cdot \left(\nabla u \Big/ \left(1 + |\nabla u|^2\right)^{1/2}\right) = 0.$$

This PDE is known as the minimal surface equation.

1. Write down the previous PDE in the case $n = 2$.

2. Show that the plane $u(x, y) = Ax + By + C$ is a (trivial) solution to this equation.

3. Show that the following are non-trivial solutions

$$u_1(x, y) = \tan^{-1}(y/x); \quad u_2(x, y) = \frac{1}{a}\cosh^{-1}\left(a\sqrt{x^2 + y^2}\right); \quad u_3(x, y) = \frac{1}{a}\log\frac{\cos ay}{\cos ax}, \tag{1.22}$$

   where a is a real constant.

4. Show that the helicoid surface (u_1) is a harmonic function. In fact, it is the is the only non-trivial solution that is a harmonic function while the catenoid (u_2) is the only non-trivial solution that is a surface of revolution and the Scherk surface (u_3) ant, is the only nontrivial solution that can be written in the form $u(x, y) = f(x) + g(y)$.